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Dynamic response of micro-periodic composite rods with uncertain parameters under moving random load

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Abstract

In this paper, axial vibrations of a finite micro-periodic composite rod with uncertain parameters under a moving random load are investigated. The solution of the problem was found by using the random dynamic influence function and applying the perturbation method. The average tolerance approach was also used to pass from differential equations with periodic coefficients to differential equations with constant coefficients. Two types of the moving random load are considered: a normal stationary or non-stationary process, the continuous load model (CLM), and a random train of moving forces, the discrete load model (DLM). Finally, a numerical example is provided to demonstrate the algorithm in the context of a computer implementation. With slight modification, the algorithm for longitudinal vibrations of a rod could also be applied to the dynamics of periodic beams, plates and shells with random parameters subjected to stochastic moving loads.

1. Introduction

Understanding the dynamic response of a structure subjected to moving loads is an interesting and important problem, with applications in bridges, roadways, railways, runways, missiles, aircraft, various machines and so on. Different types of structures and girders, such as beams, plates, shells and frames, have been considered. Different models of moving loads have also been assumed [1–3]. Both deterministic and stochastic approaches have been presented [4–14]. In most papers, parameters of the structure have been assumed to be deterministic; only a few papers consider these parameters to be random [15–17]. In these papers, the vibration of a beam supported on a random subsoil and subjected to moving load was considered. We have not found papers in which a dynamic response of a structure with uncertain parameters subjected to moving random loads has been considered. An important problem not considered yet is the dynamic response of a composite or periodic structure with uncertain parameters subjected to a moving random load. The problem of the dynamic and static response of a periodic or composite rod or beam in a deterministic

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approach has been considered, among others [18–21]. In these articles, the effective solutions were found thanks to the homogenization method. The problem of vibrations of a periodic structure caused by a moving load has also been considered [22,23].

In the paper, axial vibrations of a finite micro-periodic composite rod with uncertain parameters under a moving random load are investigated. The solution of the problem was found by using the random dynamic influence function [24,25] and applying the perturbation method. The average tolerance approach was also used to pass from differential equations with periodic coefficients to differential equations with constant coefficients. For applications of the average tolerance approach in dynamics of beams, see Refs. [29,30]. The tolerance averaging method has several advantages, and may be used as an alternative to the well-known homogenization method. The homogenization method expands the solution twice; once in terms of the small parameter and once in terms of the random variables. Thus, we decided to use the tolerance averaging method to avoid having to apply the perturbation method twice, which would be necessary within the framework of the homogenization method. Two types of the random moving load are considered, namely a normal stationary or non-stationary process, the continuous load model (CLM), and a random train of moving forces, the discrete load model (DLM).

2. General solution

Let us consider the stochastic vibrations of a periodic straight rod of length L with a varying cross-section and spatially distributed uncertain parameters, excited by a load moving with a constant velocity v as shown in Fig. 1. The differential equation for the motion of the rod has the form:

$$-[K(\mathbf{b},x)u_{,x}(\mathbf{b},x,t)]_{,x} + c(\mathbf{b},x)\dot{u}(\mathbf{b},x,t) + m(\mathbf{b},x)\ddot{u}(\mathbf{b},x,t) = p(x,\ t), \tag{1}$$

where $u(\mathbf{b},x,t)$ denotes the axial displacement, $K(\mathbf{b},x)$, $m(\mathbf{b},x)$, $c(\mathbf{b},x)$ are the uncertain axial rod rigidity, mass of the rod per unit length, damping coefficient, respectively, which are random functions of the spatial coordinate x, $x \in [0, L]$. It is assumed that the excitation process of the rod is a stochastic load, moving with a constant velocity. Additionally we assume that the structural and load parameters are independent. The susbscript x and the superscript dot denote differentiation in space and time, respectively. As a model

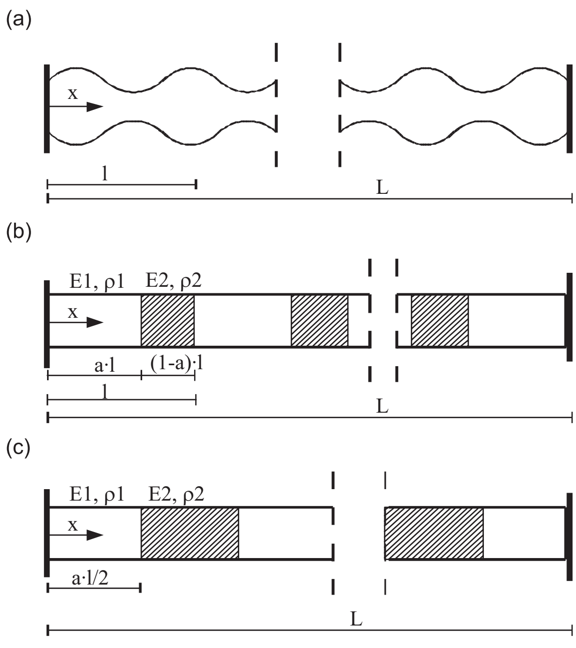

Fig. 1. Examples of periodic rods.

problem, we consider a rod composed of a periodic array of two linearly elastic, homogeneous and isotropic constituents with perfect interfaces, as illustrated in Fig. 1.

The standard methods of analyzing the rod dynamics are effective only if the coefficients in Eq. (1) are deterministic and either constant or slowly varing. The quantities $K(\boldsymbol{b}, x)$, $c(\boldsymbol{b}, x)$ and $m(\boldsymbol{b}, x)$ in this study are modeled as random periodic fields and are rapidly varying l-periodic functions:

$$K(\mathbf{b}, x) = K(\mathbf{b}, x + l), \quad c(\mathbf{b}, x) = c(\mathbf{b}, x + l), \quad m(\mathbf{b}, x) = m(\mathbf{b}, x + l). \tag{2}$$

The length l is small compared to the length L of the rod ($l \ll L$). The random parameters of the rod are presented as a vector $\mathbf{b} = [b_1, b_2, \ldots, b_r]^T$, where the superscript T denotes the transposition operation. It is assumed that the expected value $E[\mathbf{b}]$ and the covariance matrix $\mathbf{C_{bb}} = [\text{cov}(b_i, b_j)]_{rxr} = E[\mathbf{bb}^T] - E[\mathbf{b}]E[\mathbf{b}^T]$ are known. Possible random rod parameters include: the Young modulus, the damping coefficient and the dimensions of the rod cross-section. The solution will be found within correlation theory, and therefore exact knowledge of the probability distributions of these random variables is not required.

We consider two types of stochastic moving loads, namely a normal stationary or non-stationary process, the CLM, and a random train of moving forces, the DLM. In the first case (CLM) the loading process has the form $p(x,t) = p(x-vt) = p(\xi)$. It is assumed that the expected value $E[p(\xi)]$ and the covariance function $C_{pp}(\xi_1, \xi_2)$ are known and $\xi = x-vt$. In the second case (DLM) the loading process has the form:

$$p(x, t) = \sum_{k=1}^{N(t)} A_k \delta[x - v(t - t_k)], \tag{3}$$

where the amplitudes A_k are random variables, which are mutually independent and also independent from the times t_k and their expected values $E[A_k] = E[A]$, $E[A_k^2] = E[A^2]$ are known, $\delta(\cdot)$ denotes the Dirac delta function. The forces arrive at the beam at random times t_k which constitute a Poisson process $N(t)$ with the parameter λ.

The aim of the paper is to find the solution for probabilistic characteristic of the response $u(\mathbf{b}, x, t)$ of the rod. The probabilistic characteristics of the response of the rod are sought in the form of the first two probabilistic moments, i.e., the expected value and the correlation (covariance) function. Two difficulties arise in this problem, in that the coefficients in Eq. (1) are both strongly periodic and random.

When the parameters of Eq. (1) are random, the problem can be solved only if the right-hand side of Eq. (1) is deterministic. To overcome these difficulties, we introduce the random dynamic moving influence function (RDMIF) $U(\mathbf{b}, x, t)$ which satisfies the following equation:

$$-(K(\mathbf{b}, x)U_{,x}(\mathbf{b}, x, t))_{,x} + c(\mathbf{b}, x)\dot{U}(\mathbf{b}, x, t) + m(\mathbf{b}, x)\ddot{U}(\mathbf{b}, x, t) = \delta(x - vt). \tag{4}$$

Now, the right-hand side of Eq. (4) is deterministic and the response of the rod $u(\mathbf{b},x,t)$ for a stochastic load process can be expressed, for zero initial conditions for Eq. (1), by the means of RDMIF by the relationship:

$$u(\mathbf{b}, x, t) = \int_0^t U(\mathbf{b}, x, t - \tau)p(\tau)\,d\tau. \tag{5}$$

Thus, in order to determine the probabilistic characteristics of the displacement of the rod, one can apply the expectancy operator to Eq. (5) and consequently obtain the expected value:

$$E[u(\mathbf{b}, x, t)] = \int_0^t E[U(\mathbf{b}, x, t - \tau)]E[p(\tau)]\,d\tau \tag{6}$$

and the covariance of the displacement:

$$\begin{aligned}\text{Cov}_{uu}[x_1, x_2, t_1, t_2] = &\int_0^{t_1}\int_0^{t_2} E[U(\mathbf{b}, x_1, t_1 - \tau_1)U(\mathbf{b}, x_2, t_2 - \tau_2)]\text{Cov}_{pp}[\tau_1, \tau_2]\,d\tau_1\,d\tau_2 \\ &+ \int_0^{t_{1i}}\int_0^{t_2} \text{Cov}_{UU}[x_1, x_2, t_1 - \tau_1, t_2 - \tau_2]E[p(\tau_1)]E[p(\tau_2)]\,d\tau_2\,d\tau_2,\end{aligned} \tag{7}$$

where the covariance of RDIF can be estimated from:

$$\mathrm{Cov}_{UU}[x_1, x_2, t_1, t_2] = E[U(\mathbf{b}, x_1, t_1)U(\mathbf{b}, x_2, t_2)] - E[U(\mathbf{b}, x_1, t_1)]E[U(\mathbf{b}, x_2, t_2)] \tag{8}$$

and $\mathrm{Cov}_{pp}(\tau_1,\tau_2)$ denotes the time covariance of the excitation force.

In the case of a random train of moving forces (Eq. (3)), the response of the rod can be presented in the form a Duhamel–Stielties integral, as follows:

$$u(\mathbf{b}, x, t) = \int_0^t A(\tau)U(\mathbf{b}, x, t-\tau)\,\mathrm{d}N(\tau), \tag{9}$$

where $\mathrm{d}N(t)$ is the increment of the Poisson process in time $(t,\ t+\mathrm{d}t)$.

Taking into account the following properties:

$$\begin{aligned} &E[\mathrm{d}N^k(t)] = \lambda\,\mathrm{d}t \text{ for } k = 1, 2, \ldots \\ &E[\mathrm{d}N(t_1)\,\mathrm{d}N(t_2)] = \lambda^2\,\mathrm{d}t_1\,\mathrm{d}t_2 \text{ for } t_1 \neq t_2, \end{aligned} \tag{10}$$

one obtains the probabilistic characteristics of the response:

$$\begin{aligned} E[u(x,t)] &= \lambda E[A]\int_0^t E[U(b, x, t-\tau)]\,\mathrm{d}\tau \\ \mathrm{Cov}_{uu}[x_1, x_2, t_1, t_2] &= \lambda E[A^2]\int_0^{t_{\min}} E[U(b, x_1, t_1-\tau)U(b, x_2, t_2-\tau)]\,\mathrm{d}\tau \\ &+ \lambda^2 E^2[A]\int_0^{t_1}\int_0^{t_2} \mathrm{Cov}_{UU}[x_1, x_2, t_1-\tau_1, t_2-\tau_2]\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2, \end{aligned} \tag{11}$$

where $t_{\min} = \min(t_1,\ t_2)$.

Accordingly, we have obtained the formulas for second-order probabilistic moments of the displacement of the rod. The randomness of the rod parameters is accounted for in the RDMIF $U(\mathbf{b},\ x,\ t)$, which depends on the uncertain parameter vector $\mathbf{b}$. The integrals in Eqs. (6), (7) and (11) may be solved using a numerical procedure. Here, another difficulty arises in the determination of the expected values and the second moment (covariance) of the RDMIF which are in Eqs. (6), (7) and (11). This will be addressed presently.

3. The tolerance averaging approximation method

It is difficult to find the solution to Eq. (4) as the coefficients are strongly periodic. Thus, we solve Eq. (4) based on the tolerance-averaged model [26–28]. Using this procedure it is possible to transform Eq. (4) to the form of a system of averaged differential equations with constant coefficients. This approximation describes the effect of the structural length parameter of the rod. We define $\Omega = (0,L)$, $\Delta(x) = (x-(l/2),\ x+(l/2))$, $l \ll L$, $x \in \Omega^0$, $= \{x \in \Omega^0\text{: } \Delta(x) \subset \Omega\}$. The periodic functions will be averaged by means of the formula

$$\langle g(x,t)\rangle = \frac{1}{l}\int_{x-(l/2)}^{x+(l/2)} g(\xi, t)\,\mathrm{d}\xi, \quad x \in \Omega^0 \tag{12}$$

where $g(x,\ t)$ is an arbitrary function defined on $\Omega = (0,\ L)$.

We base on conformability assumption [26–28] that the function $U(\mathbf{b},\ x,\ t)$ conforms to the l-periodic structure of the rod, and together with all its derivatives has to be periodic-like. Let us introduce the following decomposition of this function:

$$U(\mathbf{b}, x, t) = W(\mathbf{b}, x, t) + V(\mathbf{b}, x, t), \tag{13}$$

where $W(\mathbf{b},\ x,\ t)$ is the averaged part of the function $U(\mathbf{b},\ x,\ t)$ and $V(\mathbf{b},\ x,\ t)$ will be referred to as the fluctuating part of the function $U(\mathbf{b},\ x,\ t)$.

The modeling decomposition Eq. (13) makes it possible to introduce two kinds of basic unknowns, namely function $W(\mathbf{b},\ x,\ t)$, which is a slowly varying function, and $V(\mathbf{b},\ x,\ t)$, which is an oscillating l-periodic-like function.

Using the Galerkin approximation we obtain the fluctuating function in the form:

$$V(\mathbf{b},x,t) = g^A(x)v^A(\mathbf{b},x,t) \tag{14}$$

(the summation convention over $A = 1, 2, \ldots$ holds), where $g^A(x)$ are a priori known oscillating l-periodic-like functions and the new unknown amplitudes $v^A(\mathbf{b}, x, t)$ are sufficiently regular and slowly varying functions.

The functions should satisfy conditions

$$\left\langle g^A(x)\right\rangle = \frac{1}{l}\int_{x-(l/2)}^{x+(l/2)} g^A(x)\,\mathrm{d}x = 0 \tag{15}$$

and

$$\left\langle m(\mathbf{b},x)g^A(x)\right\rangle = \frac{1}{l}\int_{x-(l/2)}^{x+(l/2)} m(\mathbf{b},x)g^A(x)\,\mathrm{d}x = 0. \tag{16}$$

Using the decomposition of Eqs. (13) and (14), taking into account the tolerance averaging approximation [26–28], and applying some further manipulations, we obtain the following system of $N+1$ equations with constant coefficients for unknown functions $W(\mathbf{b}, x, t)$ and $v^A(\mathbf{b}, x, t)$, for $x\in\Omega_0$

$$\begin{aligned}
&-\left\langle K(\mathbf{b},x)\right\rangle W_{,xx}(\mathbf{b},x,t) - \left\langle K(\mathbf{b},x)g^A_{,x}(x)\right\rangle v^A_{,x}(\mathbf{b},x,t) + \left\langle c(\mathbf{b},x)\right\rangle \dot{W}(\mathbf{b},x,t)\\
&\quad + \left\langle m(\mathbf{b},x)\right\rangle \ddot{W}(\mathbf{b},x,t) = \delta(x-vt),\\
&+\left\langle K(\mathbf{b},x)g^B_{,x}(x)\right\rangle W_{,x}(\mathbf{b},x,t) + \left\langle K(\mathbf{b},x)g^B_{,x}(x)g^A_{,x}(x)\right\rangle v^A(\mathbf{b},x,t)\\
&\quad + \left\langle c(\mathbf{b},x)g^B(x)g^A(x)\right\rangle \dot{v}^A(\mathbf{b},x,t) + \left\langle m(\mathbf{b},x)g^B(x)g^A(x)\right\rangle \ddot{v}^A(\mathbf{b},x,t) = 0,
\end{aligned} \tag{17}$$

where $A, B = 1, 2, \ldots, N$.

It has been assumed that the damping coefficient fulfills $c(\mathbf{b}, x) = 2\alpha m(\mathbf{b}, x)$, where α is constant, and hence $\langle c(\mathbf{b}, x)g^A(x)\rangle = 0$. The derivation of the rod Eq. (17) is analogous to the derivation of the beam equations from Ref. [29]. A derivation of Eq. (17) is presented in Appendix A.

The next aim of the paper is to find probabilistic moments of the functions $W(\mathbf{b}, x, t)$ and $v^A(\mathbf{b}, x, t)$, which describe the RDMIF $U(\mathbf{b}, x, t)$. The coefficients of the system of Eq. (17) are random but the right-hand sides are deterministic, and thus the perturbation method can be used.

4. Probabilistic characteristic of random dynamic influence function

We expand the random functions $K(\mathbf{b}, x)$, $c(\mathbf{b}, x)$, $m(\mathbf{b}, x)$ and $W(\mathbf{b}, x, t)$, $v^A(\mathbf{b}, x, t)$ into Taylor series around their expected values

$$K(\mathbf{b},x) = K^0(x) + \sum_{i=1}^r K^{\mathrm{I}}_i(x)\tilde{b}_i + \frac{1}{2}\sum_{i=1}^r\sum_{j=1}^r K^{\mathrm{II}}_{ij}(x)\tilde{b}_i\tilde{b}_j + \cdots, \tag{18}$$

$$c(\mathbf{b},x) = c^0(x) + \sum_{i=1}^r c^{\mathrm{I}}_i(x)\tilde{b}_i + \frac{1}{2}\sum_{i=1}^r\sum_{j=1}^r c^{\mathrm{II}}_{ij}(x)\tilde{b}_i\tilde{b}_j + \cdots, \tag{19}$$

$$m(\mathbf{b},x) = m^0(x) + \sum_{i=1}^r m^{\mathrm{I}}_i(x)\tilde{b}_i + \frac{1}{2}\sum_{i=1}^r\sum_{j=1}^r m^{\mathrm{II}}_{ij}(x)\tilde{b}_i\tilde{b}_j + \cdots, \tag{20}$$

$$W(\mathbf{b},x,t) = W^0(x,t) + \sum_{i=1}^r W^{\mathrm{I}}_i(x,t)\tilde{b}_i + \frac{1}{2}\sum_{i=1}^r\sum_{j=1}^r W^{\cdot}_{ij}(x,t)\tilde{b}_i\tilde{b}_j + \cdots \tag{21}$$

$$v^A(\mathbf{b},x,t) = v^{A0}(x,t) + \sum_{i=1}^r v^{A\mathrm{I}}_i(x,t)\tilde{b}_i + \frac{1}{2}\sum_{i=1}^r\sum_{j=1}^r v^{A\mathrm{II}}_{ij}(x,t)\tilde{b}_i\tilde{b}_j + \cdots, \tag{22}$$

where

$$\tilde{b}_i = b_i - E[b_i] = b_i - \bar{b}_i, \quad E[\tilde{b}_i] = 0, \quad K^0(x) = K(\bar{\mathbf{b}}, x), \quad c^0(x) = c(\bar{\mathbf{b}}, x), \quad m^0(x) = m(\bar{\mathbf{b}}, x),$$
$$W^0(x,t) = W(\bar{\mathbf{b}}, x, t), \quad v^{A0}(x,t) = v^A(\bar{\mathbf{b}}, x, t),$$
$$K_i^{\mathrm{I}}(x) = \left.\frac{\partial K(\mathbf{b},x)}{\partial b_i}\right|_{\mathbf{b}=\bar{\mathbf{b}}}, \quad c_i^{\mathrm{I}}(x) = \left.\frac{\partial c(\mathbf{b},x)}{\partial b_i}\right|_{\mathbf{b}=\bar{\mathbf{b}}}, \quad m_i^{\mathrm{I}}(x) = \left.\frac{\partial m(\mathbf{b},x)}{\partial b_i}\right|_{\mathbf{b}=\bar{\mathbf{b}}},$$
$$W_i^{\mathrm{I}}(x,t) = \left.\frac{\partial W(\mathbf{b},x,t)}{\partial b_i}\right|_{\mathbf{b}=\bar{\mathbf{b}}}, \qquad v_i^{A\mathrm{I}}(x,t) = \left.\frac{\partial v^A(\mathbf{b},x,t)}{\partial b_i}\right|_{\mathbf{b}=\bar{\mathbf{b}}},$$
$$K_{ij}^{\mathrm{II}}(x) = \left.\frac{\partial^2 K(\mathbf{b},x)}{\partial b_i\,\partial b_j}\right|_{\mathbf{b}=\bar{\mathbf{b}}}, \quad c_{ij}^{\mathrm{II}}(x) = \left.\frac{\partial^2 c(\mathbf{b},x)}{\partial b_i\,\partial b_j}\right|_{\mathbf{b}=\bar{\mathbf{b}}}, \quad m_{ij}^{\mathrm{II}}(x) = \left.\frac{\partial^2 m(\mathbf{b},x)}{\partial b_i\,\partial b_j}\right|_{\mathbf{b}=\bar{\mathbf{b}}},$$
$$W_{ij}^{\mathrm{II}}(x,t) = \left.\frac{\partial^2 W(\mathbf{b},x,t)}{\partial b_i\,\partial b_j}\right|_{\mathbf{b}=\bar{\mathbf{b}}}, \quad v_{ij}^{A\mathrm{II}}(x,t) = \left.\frac{\partial^2 v^A(\mathbf{b},x,t)}{\partial b_i\,\partial b_j}\right|_{\mathbf{b}=\bar{\mathbf{b}}}.$$

The unknown functions are: $W^0(x, t)$, $W_i^{\mathrm{I}}(x, t)$, $W_{ij}{}^{\mathrm{II}}(x, t)$ and $v^{A0}(x, t)$, $v_i{}^{A0}(x, t)$, $v_{ij}{}^{A\mathrm{II}}(x, t)$. After substituting Eqs. (18)–(22) into Eq. (17) and grouping with respect to $\tilde{b}_i$ one obtains a recurrence set of differential equations with constant coefficients:

- zeroth-order equations

$$\begin{aligned}
&-\langle K^0(x)\rangle W^0_{,xx}(x,t) - \left\langle K^0(x) g^A_{,x}(x)\right\rangle v^{A0}_{,x}(x,t) + \langle c^0(x)\rangle \dot{W}^0(x,t) \\
&+ \langle m^0(x)\rangle \ddot{W}^0(x,t) = \delta(x - vt), \\
&\left\langle K^0(x) g^B_{,x}(x)\right\rangle W^0_{,x}(x,t) + \left\langle K^0(x) g^B_{,x}(x) g^A_{,x}(x)\right\rangle v^{A0}(x,t) \\
&+ \langle c^0(x) g^B(x) g^A(x)\rangle \dot{v}^{A0}(x,t) + \langle m^0(x) g^B(x) g^A(x)\rangle \ddot{v}^{A0}(x,t) = 0.
\end{aligned} \tag{23}$$

- first-order equations (for $i = 1,2,\ldots,r$)

$$\begin{aligned}
&-\langle K^0(x)\rangle W^{\mathrm{I}}_{i,xx}(x,t) + \left\langle K^0(x) g^A_{,x}(x)\right\rangle v^{A\mathrm{I}}_{i,x}(x,t) \\
&\quad + \langle c^0(x)\rangle \dot{W}^{\mathrm{I}}_i(x,t) + \langle m^0(x)\rangle \ddot{W}^{\mathrm{I}}_i(x) = \langle K^{\mathrm{I}}_i(x)\rangle W^0_{,xx}(x,t) \\
&\quad + \left\langle K^{\mathrm{I}}_i(x) g^A_{,x}(x)\right\rangle v^{A0}_{,x}(x,t) - \langle c^{\mathrm{I}}_i(x)\rangle \dot{W}^0(x,t) - \langle m^{\mathrm{I}}_i(x)\rangle \ddot{W}^0(x,t), \\
&\left\langle K^0(x) g^B_{,x}(x)\right\rangle W^{\mathrm{I}}_{i,x}(x,t) + \left\langle K^0(x) g^B_{,x}(x) g^A_{,x}(x)\right\rangle v^{A\mathrm{I}}_i(x,t) \\
&\quad + \langle c^0(x) g^B(x) g^A(x)\rangle \dot{v}^{A\mathrm{I}}_i(x,t) + \langle m^0(x) g^B(x) g^A(x)\rangle \ddot{v}^{A\mathrm{I}}_i(x,t) \\
&= -\left\langle K^{\mathrm{I}}_i(x) g^B_{,x}(x)\right\rangle W^0_{,x}(x,t) - \left\langle K^{\mathrm{I}}_i(x) g^A_{,x}(x) g^B_{,x}(x)\right\rangle v^{A0}(x,t) \\
&\quad - \langle c^{\mathrm{I}}_i(x) g^B(x) g^A(x)\rangle \dot{v}^{A0}(x,t) - \langle m^{\mathrm{I}}_i(x) g^B(x) g^A(x)\rangle \ddot{v}^{A0}(x,t).
\end{aligned} \tag{24}$$

- second-order equations (for $i,j = 1,2,\ldots,r$)

$$\begin{aligned}
&-\langle K^0(x)\rangle W^{\mathrm{II}}_{ij,xx}(x,t) - \left\langle K^0(x) g^A_{,x}(x)\right\rangle v^{A\mathrm{II}}_{ij,x}(x,t) \\
&\quad + \langle c^0(x)\rangle \dot{W}^{\mathrm{II}}_{ij}(x,t) + \langle m^0(x)\rangle \ddot{W}^{\mathrm{II}}_{ij}(x) \\
&= +\langle K^{\mathrm{I}}_i(x)\rangle W^{\mathrm{I}}_{j,xx}(x,t) + \left\langle K^{\mathrm{I}}_j(x)\right\rangle W^{\mathrm{I}}_{i,xx}(x,t) \\
&\quad + \left\langle K^{\mathrm{II}}_{ij}(x)\right\rangle W^0(x,t) + \left\langle K^{\mathrm{I}}_i(x) g^A_{,x}(x)\right\rangle v^{A\mathrm{I}}_{j,x}(x,t) + \left\langle K^{\mathrm{I}}_j(x) g^A_{,x}(x)\right\rangle v^{A\mathrm{I}}_{i,x}(x,t) \\
&\quad + \left\langle K^{\mathrm{II}}_{ij}(x) g^A_{,x}(x)\right\rangle v^{A0}_{,x}(x,t) - \langle c^{\mathrm{I}}_i(x)\rangle \dot{W}^{\mathrm{I}}_j(x,t) - \left\langle c^{\mathrm{I}}_j(x)\right\rangle \dot{W}^{\mathrm{I}}_i(x,t)
\end{aligned}$$

$$
\begin{aligned}
&- \left\langle c^{\mathrm{II}}_{ij}(x)\right\rangle \dot{W}^0(x,t) - \langle m^{\mathrm{I}}_i(x)\rangle \ddot{W}^{\mathrm{I}}_j(x,t) - \left\langle m^{\mathrm{I}}_j(x)\right\rangle \ddot{W}^{\mathrm{I}}_i(x,t) - \left\langle m^{\mathrm{II}}_{ij}(x)\right\rangle \ddot{W}^0(x,t),\\
&\left\langle K^0(x)g^B_{,x}(x)\right\rangle W^{\mathrm{II}}_{ij,x}(x,t) + \left\langle K^0(x)g^B_{,x}(x)g^A_{,x}(x)\right\rangle v^{A\mathrm{II}}_{ij}(x,t)\\
&+ \langle c^0(x)g^B(x)g^A(x)\rangle \dot{v}^{A\mathrm{II}}_{ij}(x,t) + \langle m^0(x)g^B(x)g^A(x)\rangle \ddot{v}^{A\mathrm{II}}_{ij}(x,t)\\
= &-\left\langle K^{\mathrm{I}}_i(x)g^B_{,x}(x)\right\rangle W^{\mathrm{I}}_{j,x}(x,t) - \left\langle K^{\mathrm{I}}_j(x)g^B_{,x}(x)\right\rangle W^{\mathrm{I}}_{i,x}(x,t) - \left\langle K^{\mathrm{II}}_{ij}(x)g^B_{,x}(x)\right\rangle W^0_{,x}(x,t)\\
&- \left\langle K^{\mathrm{I}}_i(x)g^B_{,x}(x)g^A_{,x}(x)\right\rangle v^{A\mathrm{I}}_j(x,t) - \left\langle K^{\mathrm{I}}_j(x)g^B_{,x}(x)g^A_{,x}(x)\right\rangle v^{A\mathrm{I}}_i(x,t)\\
&- \left\langle K^{\mathrm{II}}_{ij}(x)g^B_{,x}(x)g^A_{,x}(x)\right\rangle v^{A0}(x,t) - \langle c^{\mathrm{I}}_i(x)g^B(x)g^A(x)\rangle \dot{v}^{A\mathrm{I}}_j(x,t)\\
&- \left\langle c^{\mathrm{I}}_j(x)g^B(x)g^A(x)\right\rangle \dot{v}^{A\mathrm{I}}_i(x,t) - \left\langle c^{\mathrm{II}}_{ij}(x)g^B(x)g^A(x)\right\rangle \dot{v}^{A0}(x,t)\\
&- \langle m^{\mathrm{I}}_i(x)g^B(x)g^A(x)\rangle \ddot{v}^{A\mathrm{I}}_i(x,t) - \left\langle m^{\mathrm{I}}_j(x)g^B(x)g^A(x)\right\rangle \ddot{v}^{A\mathrm{I}}_j(x,t)\\
&- \left\langle m^{\mathrm{II}}_{ij}(x)g^B(x)g^A(x)\right\rangle \ddot{v}^{A0}(x,t).
\end{aligned}
\tag{25}
$$

In Eq. (25) the symmetries: $K^{\mathrm{II}}_{ij}(x) = K^{\mathrm{II}}_{ji}(x)$, $c^{\mathrm{II}}_{ij}(x) = c^{\mathrm{II}}_{ji}(x)$, $m^{\mathrm{II}}_{ij}(x) = m^{\mathrm{II}}_{ji}(x)$ and $W^{\mathrm{II}}_{ij}(x,t) = W^{\mathrm{II}}_{ji}(x,t)$, $v^{A\mathrm{II}}_{ij}(x,t) = v^{A\mathrm{II}}_{ji}(x,t)$ were taken into account.

The hierarchical Eqs. (23)–(25) consists of $N+1$ Eq. (23), $r(N+1)$ Eq. (24) and $(1/2)(N+1)r(r+1)$ Eq. (25). After solving the hierarchical system of the equations, we obtain from the relationship (13), (14), (21), (22) the two first probabilistic moments of the random impulse influence function in the following forms:

$$
\begin{aligned}
E[U(\mathbf{b},x,t)] = {} & W^0(x,t) + \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} W^{\mathrm{II}}_{ij}(x,t)\mathrm{cov}(b_i,b_j) + \cdots\\
&+ g^A(x)\left[v^{A0}(x,t) + \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} v^{A\mathrm{II}}_{ij}(x,t)\mathrm{cov}(b_i,b_j) + \cdots\right]
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
E[U(\mathbf{b},x_1,t_1)U(\mathbf{b},x_2,t_2)] = {} & W^0(x_1,t_1)W^0(x_2,t_2) + \sum_{i=1}^{r}\sum_{j=1}^{r} W^{\mathrm{I}}_i(x_1,t_1)W^{\mathrm{I}}_j(x_2,t_2)\mathrm{cov}(b_i,b_j) + \cdots\\
&+ g^A(x_1)g^B(x_2)\left[v^{A0}(x_1,t_1)v^{B0}(x_2,t_2) + \sum_{i=1}^{r}\sum_{j=1}^{r} v^{A\mathrm{I}}_i(x_1,t_1)v^{B\mathrm{I}}_j(x_2,t_2)\mathrm{cov}(b_i,b_j) + \cdots\right]\\
&+ W^0(x_1,t_1)g^{A0}(x_2,t_2)v^{A0}(x_1,t_1) + W^0(x_2,t_2)g^{B0}(x_1,t_1)v^{B0}(x_2,t_2) + \cdots,
\end{aligned}
\tag{27}
$$

where $\mathrm{cov}(b_i,b_j) = E[\tilde{b}_i,\tilde{b}_j]$.

The perturbation method in which the Taylor series expansion is restricted to two or three terms is frequently applied in the stochastic finite element method [31,32] and gives a good estimate of the desired unknown quantities if the variation coefficients of the construction parameters are smaller than 0.2.

5. Example: numerical results

As a model problem, we consider a composite rod composed of an array of two linearly elastic, homogeneous, and isotropic constituents with perfect interfaces, as illustrated in Fig. 1c. We assume that the Young moduli $E(x)$ are random variables and are equal to $b_1 = E_1$ on $(0, a)$ and $b_2 = E_2$ on (a, l) $(r = 2)$. The random variables E_1 and E_2 are assumed to be mutually independent. The cross-section area of the rod is constant and equal to F. One introduces only one $(N = 1)$ shape function $g^1(x)$, which is piecewise linear, as is shown in Fig. 3. In the presented example it is assumed that $a = l/2$. Let us consider only zeroth- and first-order approximation (Eqs. (23) and (24)):

- zeroth-order equations

$$-\frac{(\bar{E}_1+\bar{E}_2)}{(\rho_1+\rho_2)}W^0_{,xx}(x,t)-\frac{4(\bar{E}_2-\bar{E}_1)}{(\rho_1+\rho_2)}v^{10}_{,x}(x,t)+2\alpha\dot{W}^0(x,t)+\ddot{W}^0(x,t)=\frac{2}{F(\rho_1+\rho_2)}\delta(x-vt),$$
$$\frac{12(\bar{E}_2-\bar{E}_1)}{l^2(\rho_1+\rho_2)}W^0_{,x}(x,t)+\frac{48(\bar{E}_1+\bar{E}_2)}{l^2(\rho_1+\rho_2)}v^{10}(x,t)+2\alpha\dot{v}^{10}(x,t)+\ddot{v}^{10}(x,t)=0. \tag{28}$$

- first-order equations (for $i = 1,2$)

$$-\frac{\bar{E}_1+\bar{E}_2}{\rho_1+\rho_2}W^{\mathrm{I}}_{i,xx}(x,t)-\frac{4(\bar{E}_2-\bar{E}_1)}{(\rho_1+\rho_2)}v^{1\mathrm{I}}_{i,x}(x,t)+2\alpha\dot{W}^{\mathrm{I}}_i(x,t)+\ddot{W}^{\mathrm{I}}_i(x,t)=\frac{1}{\rho_1+\rho_2}R_{1i}(x,t),$$
$$\frac{12(\bar{E}_2-\bar{E}_1)}{l^2(\rho_1+\rho_2)}W^{\mathrm{I}}_{i,x}(x,t)+\frac{48(\bar{E}_1+\bar{E}_2)}{l^2(\rho_1+\rho_2)}v^{1\mathrm{I}}_i(x,t)+2\alpha\dot{v}^{1\mathrm{I}}_i(x,t)+\ddot{v}^{1\mathrm{I}}_i(x,t)=\frac{1}{\rho_1+\rho_2}R_{2i}(x,t), \tag{29}$$

where

$$R_{11}(x,t)=W^0_{,xx}(x,t)-4v^{10}_{,x}(x,t),\quad R_{21}(x,t)=\frac{12}{l^2}W^0_{,x}(x,t)-\frac{48}{l^2}v^{10}(x,t),$$
$$R_{12}(x,t)=W^0_{,xx}(x,t)+4v^{10}_{,x}(x,t),\quad R_{22}(x,t)=-\frac{12}{l^2}W^0_{,x}(x,t)-\frac{48}{l^2}v^{10}(x,t).$$

The bar over a letter denotes the expected value. The symbols ρ_1 and ρ_2 denote mass density.

The boundary conditions for the rod clamped on both edges have the form:

$$U(b,0,t)=U(b,L,t)=0. \tag{30}$$

We look for solutions of the system of Eqs. (28) and (29) in the form:

$$W^0(x,t)=\sum_{n=1}^{\infty}y^0_n(t)\sin\frac{n\pi x}{L},$$
$$v^{10}(x,t)=\sum_{n=1}^{\infty}z^{10}_n(t)\cos\frac{n\pi x}{L} \tag{31}$$

and for $i = 1, 2$

$$W^{\mathrm{I}}_i(x,t)=\sum_{n=1}^{\infty}y^{\mathrm{I}}_{in}(t)\sin\frac{n\pi x}{L},$$
$$v^{1\mathrm{I}}_i(x,t)=\sum_{n=1}^{\infty}z^{1\mathrm{I}}_{in}(t)\cos\frac{n\pi x}{L}, \tag{32}$$

where $\bar{c}^2=\left(\bar{E}_1+\bar{E}_2\right)/\left(\bar{\rho}_1+\bar{\rho}_2\right)$ is the average velocity of the wave in the rod.

Remark. Notice that in the case of a uniform rod ($E_1 = E_2 = E = b_1$), $\rho_1 = \rho_2 = \rho$ and Eqs. (23) and (24) give rise to the correct system of equations

- zeroth-order equation

$$-\frac{\bar{E}}{\rho}W^0_{,xx}(x,t)+2\alpha\dot{W}^0(x,t)+\ddot{W}^0(x,t)=\frac{1}{F\rho}\delta(x-vt), \tag{33}$$

- first-order equation

$$-\frac{\bar{E}}{\rho}W^{\mathrm{I}}_{1,xx}(x,t)+2\alpha\dot{W}^{\mathrm{I}}_{1}(x,t)+\ddot{W}^{\mathrm{I}}_{1}(x,t)=\frac{1}{\rho}W^{0}_{,xx}. \tag{34}$$

As an illustrative example related to the formulas derived in the previous sections, let us examine the probabilistic characteristics of a RDMIF for a composite rod consisting of periodic segments, each consisting of two alternating materials. The mass densities are taken as deterministic parameters given by $\bar{\rho}_1 = 7000\,\mathrm{kg/m^3}$ and $\bar{\rho}_2 = 6000\,\mathrm{kg/m^3}$. The moduli of elasticity are assumed to be random variables with the mean values $\bar{E}_1 = 12\times10^{10}\,\mathrm{Pa}$ and $\bar{E}_2 = 12\times10^{9}\,\mathrm{Pa}$. The length of the rod is equal to $L = 1\,\mathrm{m}$ and the length of the segment is $l = 0.1\,\mathrm{m}$ which results in 10 segments along the length of the rod. Cross-sectional area of the beam is taken as $F = 0.0001\,\mathrm{m^2}$. The considered composite rod under the moving force is shown in Fig. 2. In Fig. 3, one representative segment of the rod and the shape function taken for the numerical calculations are presented.

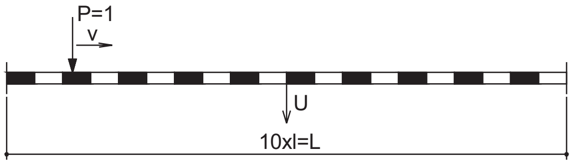

Fig. 2. The considered composite rod under a moving force.

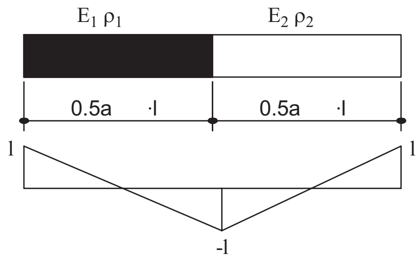

Fig. 3. The shape function for the considered composite periodic rod.

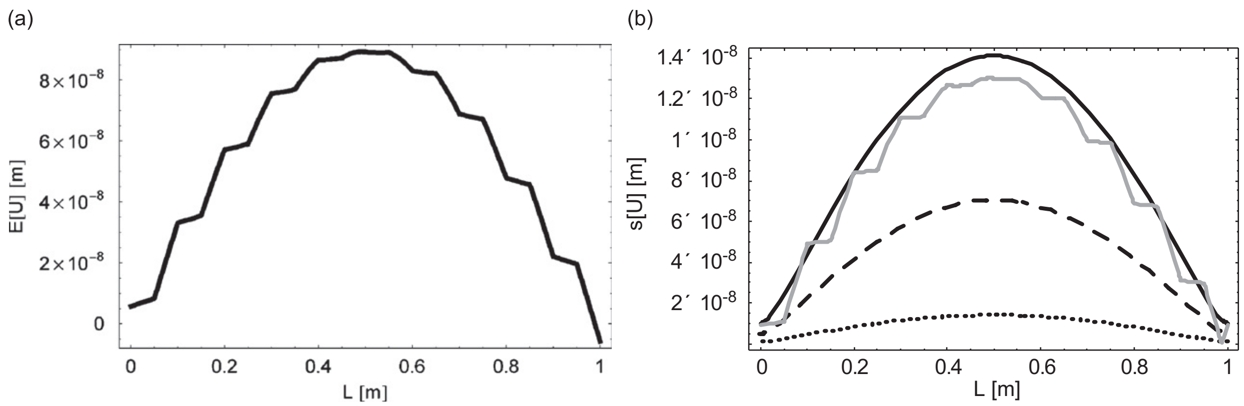

Fig. 4. Probabilistic characteristics of RDMIF U of the rod—the expected value $E[U]$ on (a) and the standard deviation $\sigma[U]$ on (b) for three coefficients of variation, namely 1% (dotted line), 5% (dashed line), and 10% (solid line) at the time instant $0.5L/v$. The gray line was obtained by the method of many realizations.

Let U denote the RDMIF. The expected value (in Fig. 4a) and the standard deviation (in Fig. 4b) of U for the three values of the coefficient of variation of the elasticity moduli E_1 and E_2, namely for 1%, 5% and 10% at the time $t = 0.5(L/v)$ were calculated. The velocity of the load is equal to $v = 0.1\bar{c}$. The results obtained using the perturbation method have been verified by the method of many realizations. The Young moduli E_1 and E_2 have been assumed to be random variables with a uniform distribution and variation coefficient equal to 0.1. The differences from the expected value $E[U]$ for both methods are less then 0.1% and charts for both methods in Fig. 4a are indistinguishable. Some differences are present on the charts for variance, but these are less than 2%.

Figs. 5a and b show the expected value and the standard deviation of the displacement U (RDMIF) at the midpoint of the rod, i.e. $x = 0.5L$, as a function of time t taken in the range (0, L/v) for the three different coefficients of variation. The velocity of the load is equal to $v = 0.1\bar{c}$.

In Figs. 6a, b, 7a, b, 8a and b, the expected value and the standard the deviation of displacement U (RDMIF) at the midpoint of the rod, i.e. $x = 0.5L$, as a function of time t taken in the range (0, L/v) are shown for three different values of the velocity of the moving force, namely 3%, 30% and 90% of the average velocity $\bar{c}$ of the wave in the rod ($v = 0.03\bar{c}$, $0.3\bar{c}$ and $0.9\bar{c}$). The standard deviation of U is shown for three values of the coefficient of variation of the moduli of elasticity E_1 and E_2, namely for 1%, 5% and 10%.

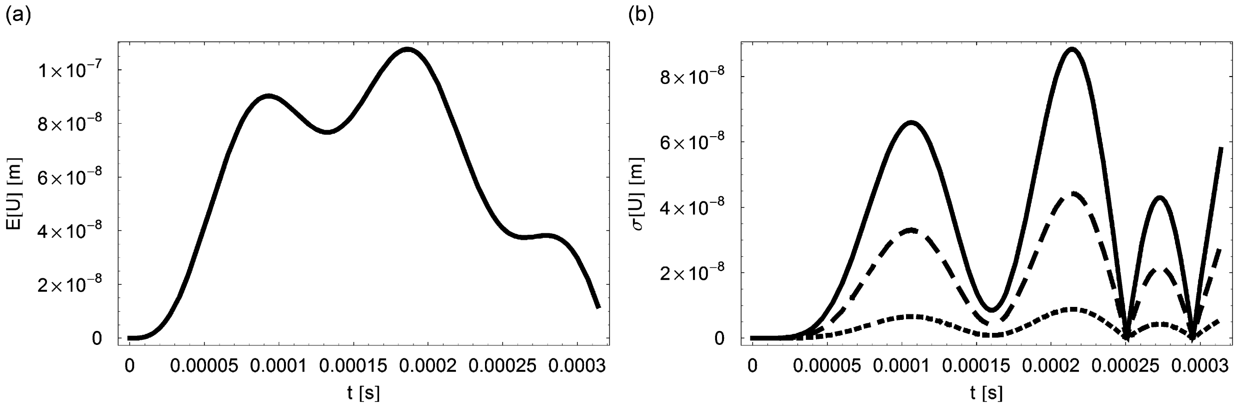

Fig. 5. Probabilistic characteristics of RDMIF U at the midpoint of the rod ($0.5L$)—the expected value $E[U]$ on (a) and the standard deviation $\sigma[U]$ on (b) for three coefficients of variation, namely 1% (dotted line), 5% (dashed line), and 10% (solid line) as a function of time $0<t<L/v$.

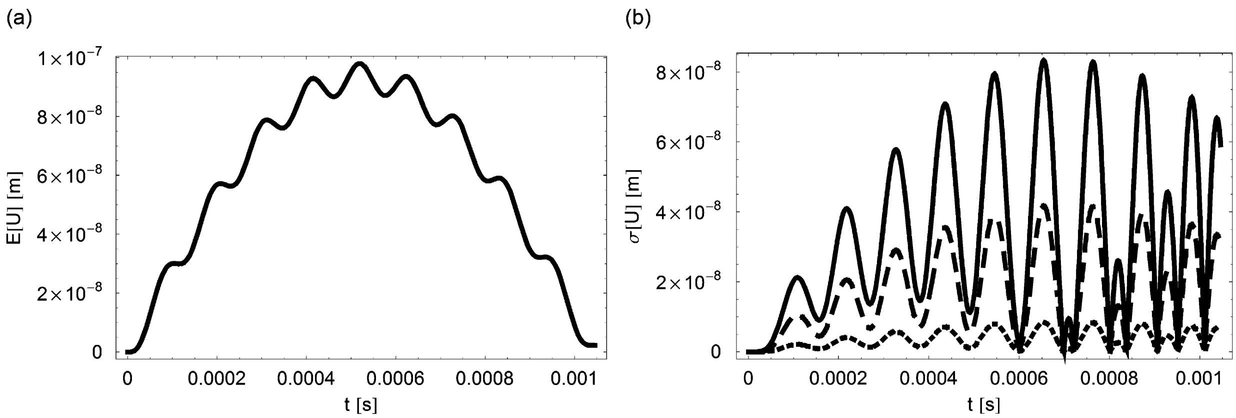

Fig. 6. Probabilistic characteristics of RDMIF U at the midpoint of the rod ($0.5L$)—the expected value $E[U]$ on (a) and the standard deviation $\sigma[U]$ on (b) as the result of a force moving with the velocity equal to $v = 0.03\bar{c}$ for three c.o.v., namely 1% (dotted line), 5% (dashed line), and 10% (solid line) as the function of time $0<t<L/v$.

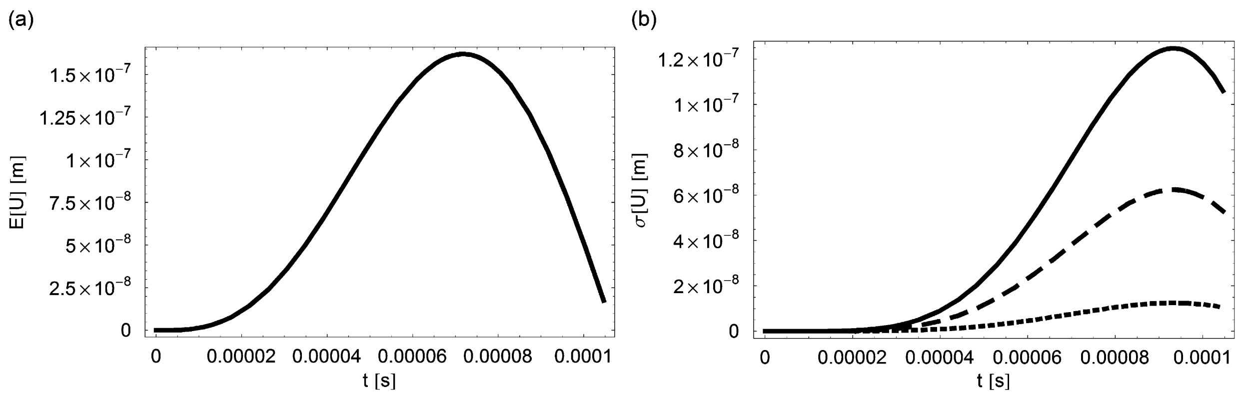

Fig. 7. Probabilistic characteristics of RDMIF U at the midpoint of the rod ($0.5L$)—the expected value $E[U]$ on (a) and the standard deviation $\sigma[U]$ on (b) as the result of a force moving with the velocity equal to $v = 0.3\bar{c}$ for three c.o.v., namely 1% (dotted line), 5% (dashed line), and 10% (solid line) as a function of time $0<t<L/v$.

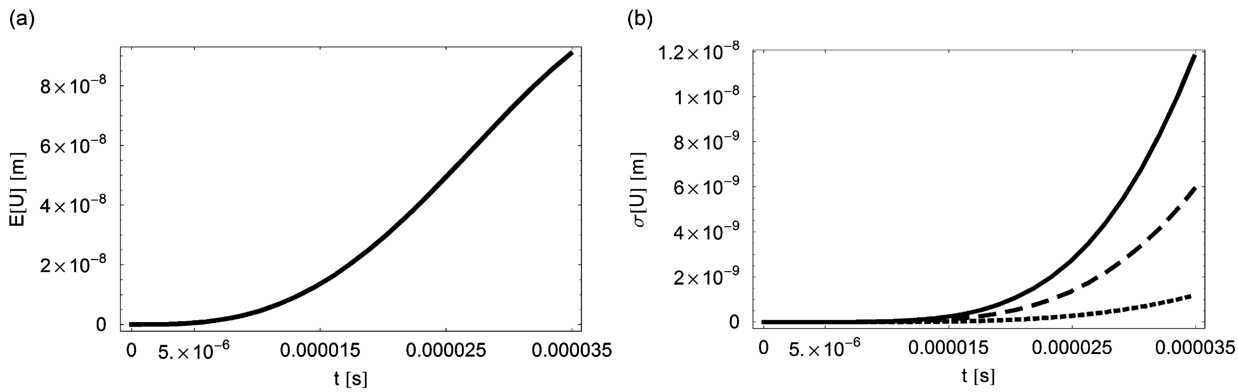

Fig. 8. Probabilistic characteristics of RDMIF U at the midpoint of the rod ($0.5L$)—the expected value $E[U]$ on (a) and the standard deviation $\sigma[U]$ on (b) as the result of a force moving with the velocity equal to $v = 0.9\bar{c}$ for three c.o.v., namely 1% (dotted line), 5% (dashed line), and 10% (solid line) as a function of time $0<t<L/v$.

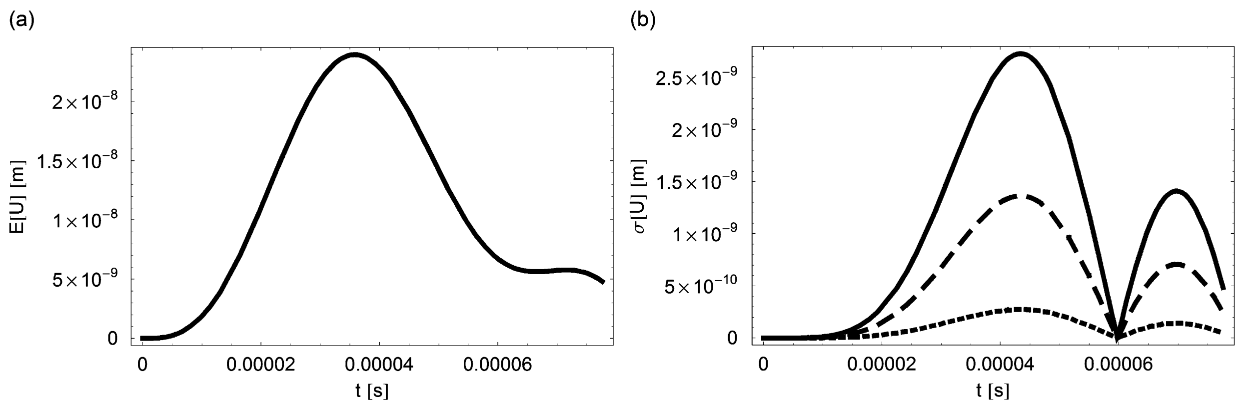

Fig. 9. Probabilistic characteristics of RDMIF U at the midpoint of the rod ($0.5L$)—the expected value $E[U]$ on (a) and the standard deviation $\sigma[U]$ on (b) for the ratio of mean values $E_2/E_1 = 1$ and for three c.o.v.:1% (dotted line), 5% (dashed line), and 10% (continuous line) as a function of time $0<t<L/v$.

In Figs. 9a, b, 10a and b, the expected value and the standard deviation of the displacement U (RDMIF) at the midpoint of the rod, i.e. $x = 0.5L$, as a function of time t taken in the range (0, L/v) are shown for two different ratios of the mean values of E_1 and E_2 ($E_2 = E_1$ and $E_2 = 0.01E_1$). The standard deviation of U is shown for three values of the coefficient of variation of the moduli of elasticity E_1 and E_2, namely for 1%, 5% and 10%. The velocity of the load is equal to $v = 0.3\bar{c}$.

Let us consider a composite periodic rod subject to a train of random forces moving with a constant velocity that constitute a Poisson stochastic process with intensity parameter λ. The forces are modeled by random variables with the probabilistic characteristics $E[A] = 1\,\mathrm{N}$ and the standard deviation $\sigma[A] = 0.1\,\mathrm{N}$. The considered composite rod under the train of moving forces is shown in Fig. 11.

In Figs. 12a and b the probabilistic characteristics of the displacement u of the rod at the middle of the rod for three values of the intensity parameter λ of Poisson's process, namely 0.05[1/s] (dotted line), 0.1[1/s] (dashed line), 0.5[1/s] (solid line) in time $0 \ll L/v$ are shown. The velocity of the load is equal to $v = 0.2\bar{c}$.

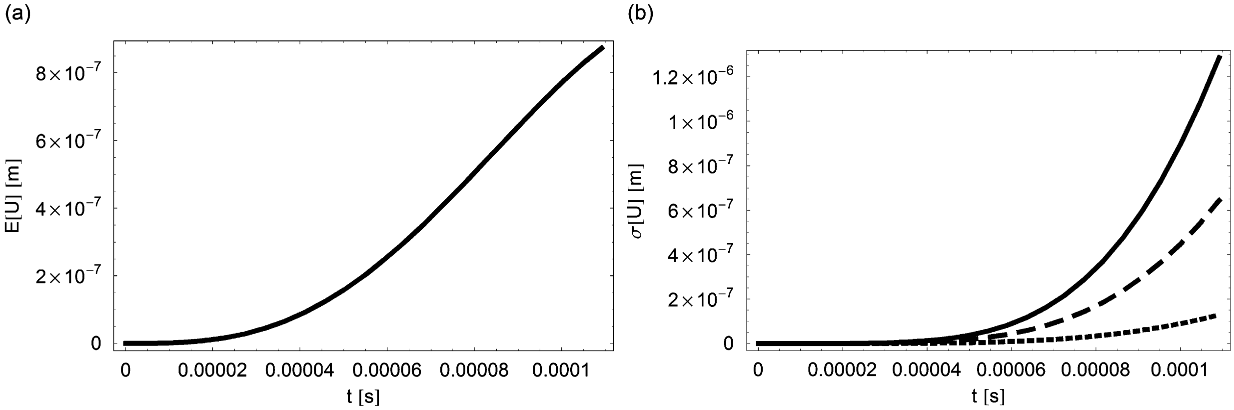

Fig. 10. Probabilistic characteristics of RDMIF U at the midpoint of the rod ($0.5L$)—the expected value $E[U]$ on (a) and the standard deviation $\sigma[U]$ on (b) for the ratio of mean values $E_2/E_1 = 0.01$ and for three c.o.v.: 1% (dotted line), 5% (dashed line), and 10% (solid line) as a function of time $0 < t < L/v$.

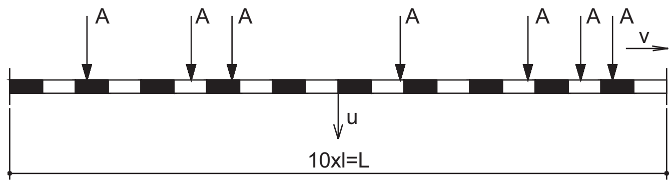

Fig. 11. The composite rod under a train of moving forces.

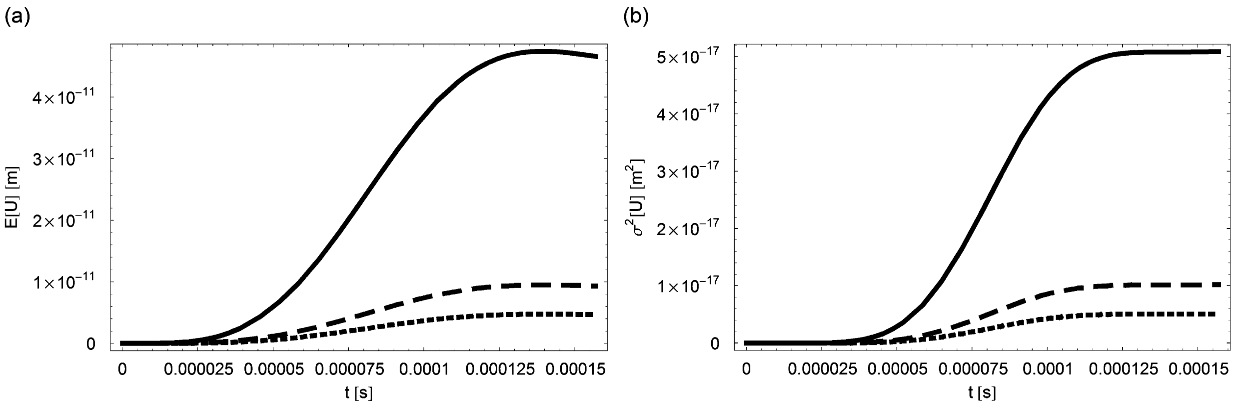

Fig. 12. Probabilistic characteristics of the displacement u of the rod at the midpoint of the rod ($0.5L$)—the expected value $E[u]$ on (a) and the variance $\sigma^2[u]$ on (b) for three values of the intensity parameter λ of Poisson's process, namely 0.05[1/s] (dotted line), 0.1[1/s] (dashed line), and 0.5[1/s] (solid line) in time $0 < t < L/v$.

In Fig. 13 the probabilistic characteristics of the displacement u of the rod, namely the expected value $E[u]$ (in Fig. 13a) and the variance $\sigma^2[u]$ (in Fig. 13b) for values of the intensity parameter λ of Poisson's process, namely 0.05[1/s] (dotted line), 0.1[1/s] (dashed line), 0.5[1/s] (solid line) at the time instant $0.5L/v$ are shown. The velocity of the load is equal to $v = 0.2\bar{c}$.

In Fig. 14 displacement U of the rod at the midpoint obtained using FEM (solid line) and average tolerance approach (dashed line) have been presented for a force moving with a constant velocity for parameters of the rod assumed to be deterministic. Realizations of the displacement obtained by the above methods are similar, whereas FEM predicts larger maximal displacements. The effective solution in FEM for 20 segments was found from the eigentransformation.

The expected values and variances of the rod response when the load process is a stationary "white noise" process (CLM model of the load) are analogous to the results for DLM load model presented in the Figs. 11–13 up to the other constants describing this process.

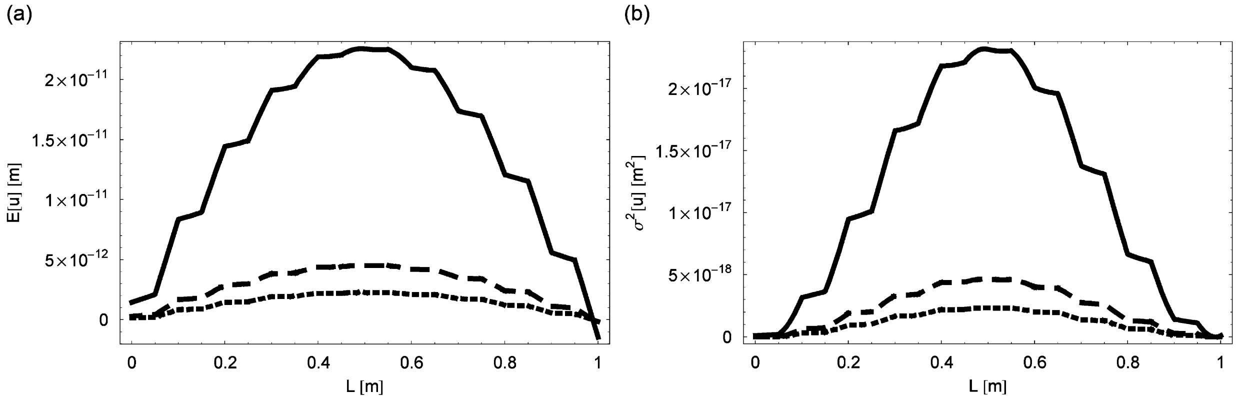

Fig. 13. Probabilistic characteristics of the displacement u of the rod—the expected value $E[u]$ on (a) and the variance $\sigma^2[u]$ on (b) for three values of the intensity parameter λ of Poisson's process, namely 0.05[1/s] (dotted line), 0.1[1/s] (dashed line), and 0.5[1/s] (solid line) at the time instant $0.5L/v$.

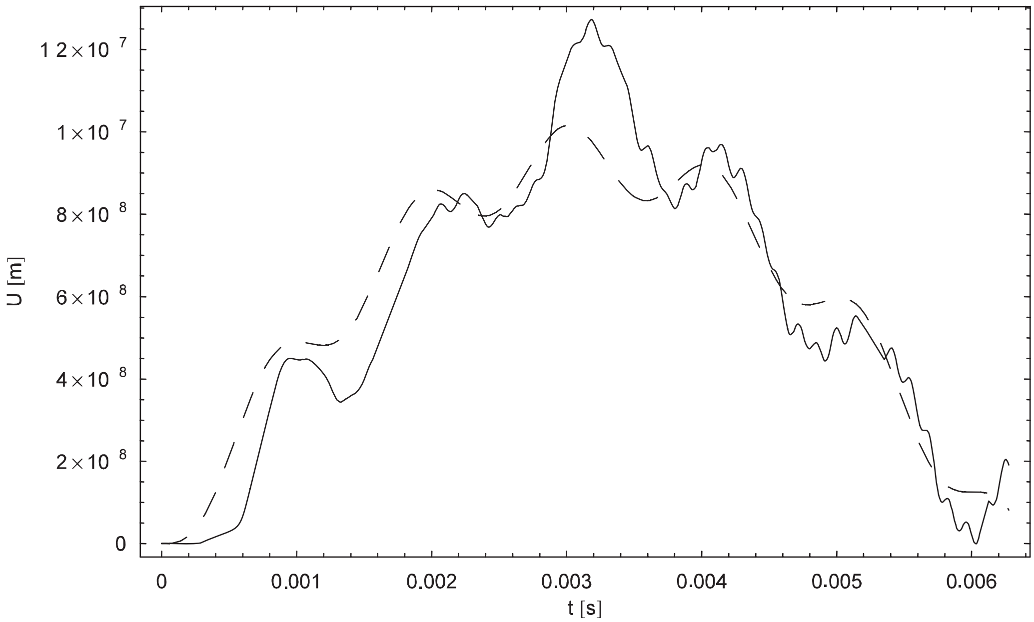

Fig. 14. Displacement U of the rod at the midpoint obtained using FEM (solid line) and tolerance averaging approach method (dashed line).

6. Conclusions

In this paper, axial vibrations of a finite micro-periodic composite rod with uncertain parameters under a moving random load are investigated. The solution of the problem was found by using the random dynamic influence function and applying the perturbation method. The average tolerance approach was also used to pass from differential equations with periodic coefficients to differential equations with constant coefficients.

Two types of moving random load were considered. The presented algorithm contains the general solution for the mean value and the correlation function of the response of a beam with uncertain parameters loaded by a moving stochastic load. The algorithm also contains particular solutions which often occur as separate scientific problems. These include solutions for vibrations of a periodic rod with random parameters caused by a force moving with a constant velocity, vibrations of a periodic rod with deterministic parameters caused by a force moving with a constant velocity, and vibrations of a periodic rod with deterministic parameters caused by a stochastic moving load.

Acknowledgement

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Appendix A

In this appendix we present a derivation of the equations in (17).

The virtual work principle for longitudinal rod vibrations has the form

$$\int_0^L N(x,t)\delta U_{,x}(x)\,\mathrm{d}x = \int_0^L [p(x,t) - c(x)\dot{U}(x,t) - m(x)\ddot{U}(x,t)]\delta U(x)\,\mathrm{d}x - N_0\delta U(0) + N_L\delta U(L), \tag{A.1}$$

where the axial force $N(x,\ t)$ is given by the formula

$$N(x,t) = K(x)U_{,x}(x,t) \tag{A.2}$$

and $p(x,\ t) = -\delta(x-vt)$.

Function $U(x,\ t)$ is given by Eqs. (13) and (14). Because the functions $v^A(x,\ t)$ are slowly varying in contrast to the oscillating functions $g^A(x)$, it follows that the derivatives with respect to the x variable are given by

$$U_{,x}(x,t) = W_{,x}(x,t) + g^A_{,x}(x)v^A(x,t). \tag{A.3}$$

By averaging the left-hand side of Eq. (A.1) and taking into account Eqs. (13), (14), (A.2) and (A.3) we obtain

$$\begin{aligned}
\int_0^L N(x,t)\delta U_{,x}(x)\,\mathrm{d}x &= \int_0^L K(x)\Big[W_{,x}(x,t) + g^A_{,x}(x)v^A(x,t)\Big]\Big[\delta W_{,x}(x) + g^B_{,x}(x)\delta v^B(x)\Big]\,\mathrm{d}x \\
&= \sum_{k=0}^{R-1}\int_{kl}^{(k+1)l} K(x)\Big[W_{,x}(x,t) + g^A_{,x}(x)v^A(x,t)\Big]\Big[\delta W_{,x}(x) + g^B_{,x}(x)\delta v^B(x)\Big]\,\mathrm{d}x \\
&= \sum_{k=0}^{R-1}\Big[\langle K(kl)\rangle W_{,x}(kl,t) + \Big\langle K(kl)g^A_{,x}(kl)v^A(kl,t)\Big\rangle\Big]\delta W_{,x}(kl) + \Big[\Big\langle K(kl)g^B_{,x}(kl)\Big\rangle W_{,x}(kl,t) \\
&\quad + \Big\langle K(kl)g^A_{,x}(kl)g^B_{,x}(kl)\Big\rangle v^A(kl)\Big]\delta v^B_{,x}(kl) \\
&\cong \int_0^L \Big\{\Big[\langle K(x)\rangle W_{,x}(x,t) + \Big\langle K(x)g^A_{,x}(x)v^A(x,t)\Big\rangle\Big]\delta W_{,x}(x) + \Big[\Big\langle K(x)g^B_{,x}(x)\Big\rangle W_{,x}(x,t) \\
&\quad + \Big\langle K(x)g^A_{,x}(x)g^B_{,x}(x)\Big\rangle v^A_{,x}(x,t)\Big]\delta v^B_{,x}(x)\Big\}\,\mathrm{d}x.
\end{aligned} \tag{A.4}$$

Similarly, way we can average the RHS of Eq. (A.1)

$$\begin{aligned}
&\int_0^L \left[p(x,t) - c(x)\dot{U}(x,t) - m(x)\ddot{U}(x,t)\right]\delta U(x)\,\mathrm{d}x - N_0\delta U(0) + N_L\delta U(L)\\
&= \int_0^L \left\{p(x,t) - c(x)\left[\dot{W}(x,t) + g^A(x)\dot{v}^A(x,t)\right] - m(x)\left[\ddot{W}(x,t) + g^A(x)\ddot{v}^A(x,t)\right]\right\}[\delta W(x)\\
&\quad + g^B(x)\delta v^B(x)\big]\,\mathrm{d}x - N_0\left[\delta W(0) + g^B(0)\delta v^B(0)\right] + N_L\left[\delta W(L) + g^B(L)\delta v^B(L)\right]\\
&= \int_0^L \left\{\left[\langle p(x,t)\rangle - \langle c(x)\rangle\dot{W}(x,t) - \langle c(x)g^A(x)\rangle\dot{v}^A(x,t) - \langle m(x)\rangle\ddot{W}(x,t)\right.\right.\\
&\quad \left. - \langle m(x)g^A(x)\rangle\ddot{v}^A(x,t)\right]\delta W(x) + \left[\langle p(x,t)g^B(x)\rangle - \langle c(x)g^B(x)\rangle\dot{W}(x,t)\right.\\
&\quad - \langle c(x)g^A(x)g^B(x)\rangle\dot{v}^A(x,t) - \langle m(x)g^B(x)\rangle\ddot{W}(x,t)\\
&\quad \left.\left. - \langle m(x)g^A(x)g^B(x)\rangle\ddot{v}^A(x,t)\right]\delta v^B(x)\right\}\mathrm{d}x. \qquad \text{(A.5)}
\end{aligned}$$

After integrating Eq. (A.4) by parts and grouping the terms with respect to the variance in Eq. (A.4) and (A.5), we obtain finally Eq. (17).

Remark. For simplicity, in the above formulas we skipped the random vector **b**.

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